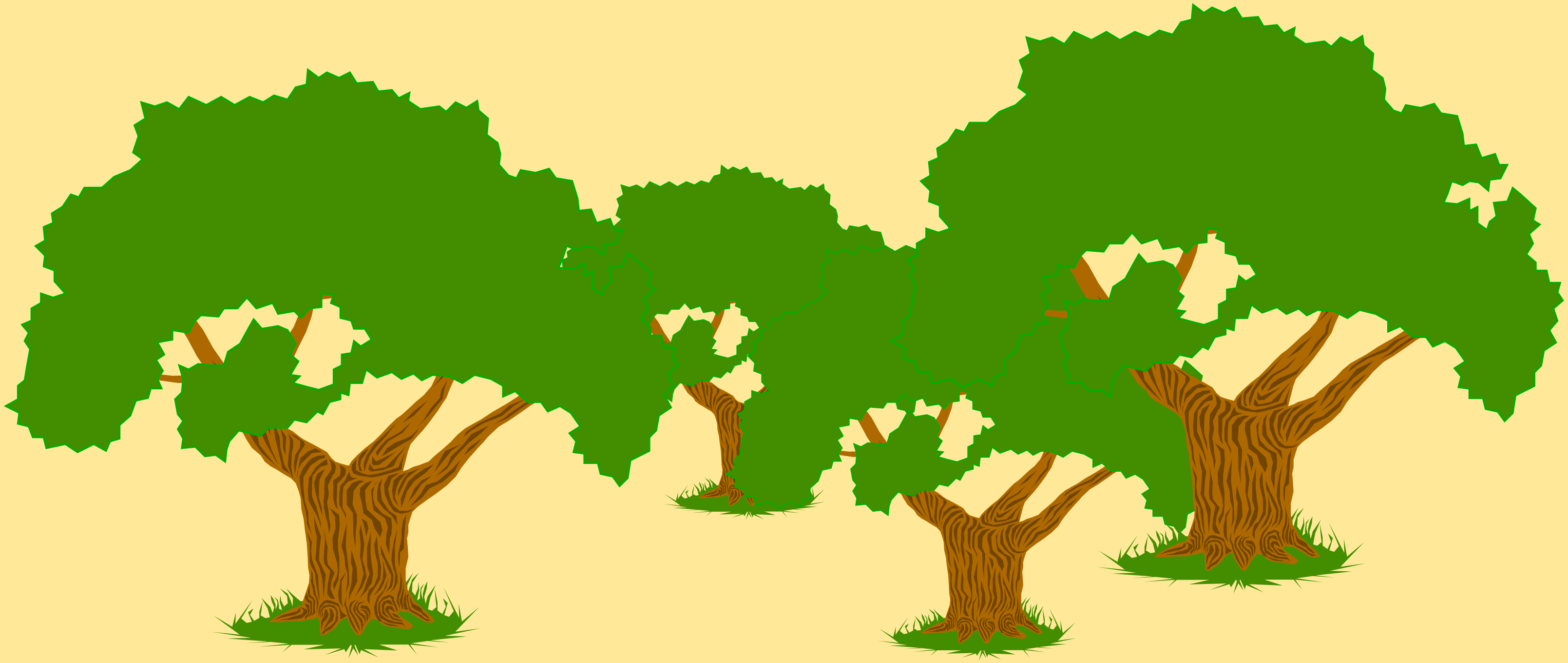
Growth functions



Summary

- Empirical *versus* biologically based growth functions
- Theoretical growth functions:
  - ✓ Lundqvist-Korf type
  - ✓ Richards type
  - ✓ Hossfeld IV type
  - ✓ Other growth functions
- (Zeide decomposition of growth functions)
- Formulating growth functions without age explicit
- Simultaneous modeling of several individuals (Families of growth functions)
  - ✓ Growth functions formulated as difference equations - ADA, (GADA and mixed models)

Growth functions

- The selection of functions – *growth functions* - appropriate to model tree and stand growth is an essencial stage in the development of growth models

  - ✓ Differencial form

$$\frac{dy}{dt} = f(t)$$

  - ✓ Integral form

$$y = \int f(t)\,dt$$

Growth functions

- Growth functions must have a shape that is in accordance with the principles of biological growth:
  - ✓ The curve is limited by yield 0 at the start (t=0 ou t=t_0) and by a maximum yield at an advanced age (existence of assymptote)
  - ✓ the relative growth rate (variation of the x variable per unit of time and unit of x) presents a maximum at a very early stage, decreasing afterwards; in most cases, the maximum occurs very early so that we can use decreasing functions to model relative growth rate
  - ✓ The slope of the curve increases in the initial stage and decreases after a certain point in time (existence of an inflexion point)

Growth functions

- Two types of functions have been used to model growth:
  - ✓ *Empirical growth functions*
    - Relationship between the dependent variable – the one we want to model – and the regressors according to some mathematical function – e.g. linear, parabolic, without trying to identify the causes or explaining the phenomenon
  - ✓ *Analitical or functional growth functions*
    - Conceived in terms of the mechanism of the system, usually having an underlying hypothesis associated with the cause or function of the phenomenon described by the response variable
- The distinction between the two is not sharp and most modeling applications contain both empiricism and mechanism in varying mixtures

Theoretical growth functions

- Theoretical growth functions have commonly been developed in their growth form – either absolute or relative growth – and the respective yield form has been obtained by integration
- Generally this approach allows interpretation of the function parameters and helps to impose restrictions on the values that the parameters can take to be biologically consistent
- Theoretical growth functions are grouped according to their functional form in:
  - ✓ Lundqvist-Korf type
  - ✓ Richards type
  - ✓ Hossfeld IV type
  - ✓ Other growth functions

Theoretical growth functions

Lundqvist-Korf type

Schumacher function

- Differential form:
  - ✓ The model proposed by Schumacher is based on the hypothesis that the relative growth rate has a linear relationship with the inverse of time^2 (which means that it decreases nonlinearly with time):

$$\frac{1}{Y}\frac{dY}{dt} = k\frac{1}{t^2} \qquad \Leftrightarrow$$

where:

- Y is the quantity of interest,
- t is time,
- k is a constant.

Schumacher function

Rearranging the equation to separate variables:

$$\frac{dY}{Y} = k \cdot \frac{1}{t^2}\,dt$$

Integrating both sides:

$$\int \frac{dY}{Y} = k \int \frac{1}{t^2}\,dt$$

The left side gives:

$$\ln|Y| = k \int t^{-2}\,dt$$

For the right side, the integral of t^{-2} is:

$$\int t^{-2}\,dt = -\frac{1}{t} + C$$

where C is the integration constant. Therefore, the equation becomes:

$$\ln|Y| = -\frac{k}{t} + C$$

Schumacher function

Exponentiate both sides to isolate Y:

$$Y(t) = e^{-\frac{k}{t}+C} = e^{C} \cdot e^{-\frac{k}{t}}$$

Let $A = e^{C}$, a positive constant. Then, the solution is:

$$Y(t) = A \cdot e^{-\frac{k}{t}}$$

Schumacher function

- Integral form:

$$y = A \; e^{-k\frac{1}{t}} \qquad\qquad A = Y_0 e^{k/t_0}$$

  - ✓ where the *A* parameter is the assymptote and (t_0, Y_0) is the initial value
  - ✓ the *k* parameter is inversely related with the growth rate

Lundqvist-Korf function

- Differential form:
  - ✓ Lundqvist-Korf is a generalization of Schumacher function with the following differential forms:

$$\frac{1}{Y}\frac{dY}{dt} = k\frac{m}{t^{(m+1)}}$$

where:

- Y is the quantity of interest,
- t is time,
- k and m are constants.

Lundqvist-Korf function

Step 1: Separate Variables

Rewriting the equation to separate the variables:

$$\frac{dY}{Y} = k \cdot \frac{m}{t^{m+1}}\, dt$$

Lundqvist-Korf function

Step 2: Integrate Both Sides

Integrate both sides to solve for Y:

$$\int \frac{dY}{Y} = k \cdot m \int t^{-(m+1)}\,dt$$

The left side becomes:

$$\ln|Y| = k \cdot m \int t^{-(m+1)}\,dt$$

The right side is integrated as follows:

$$\int t^{-(m+1)}\,dt = \int t^{-(m+1)}\,dt = \frac{t^{-(m+1)+1}}{-(m+1)+1} + C = -\frac{1}{m} \cdot t^{-m} + C$$

where C is the integration constant. Thus, the equation becomes:

$$\ln|Y| = k \cdot m \left(-\frac{1}{m} \cdot t^{-m}\right) + C$$

Lundqvist-Korf function

Simplify:

$$\ln|Y| = -k \cdot t^{-m} + C$$

Step 3: Solve for Y

Exponentiate both sides to isolate Y:

$$Y(t) = e^{-k \cdot t^{-m} + C} = e^{C} \cdot e^{-k \cdot t^{-m}}$$

Let $A = e^{C}$, a positive constant, to obtain the integral form:

$$Y(t) = A \cdot e^{-k \cdot t^{-m}}$$

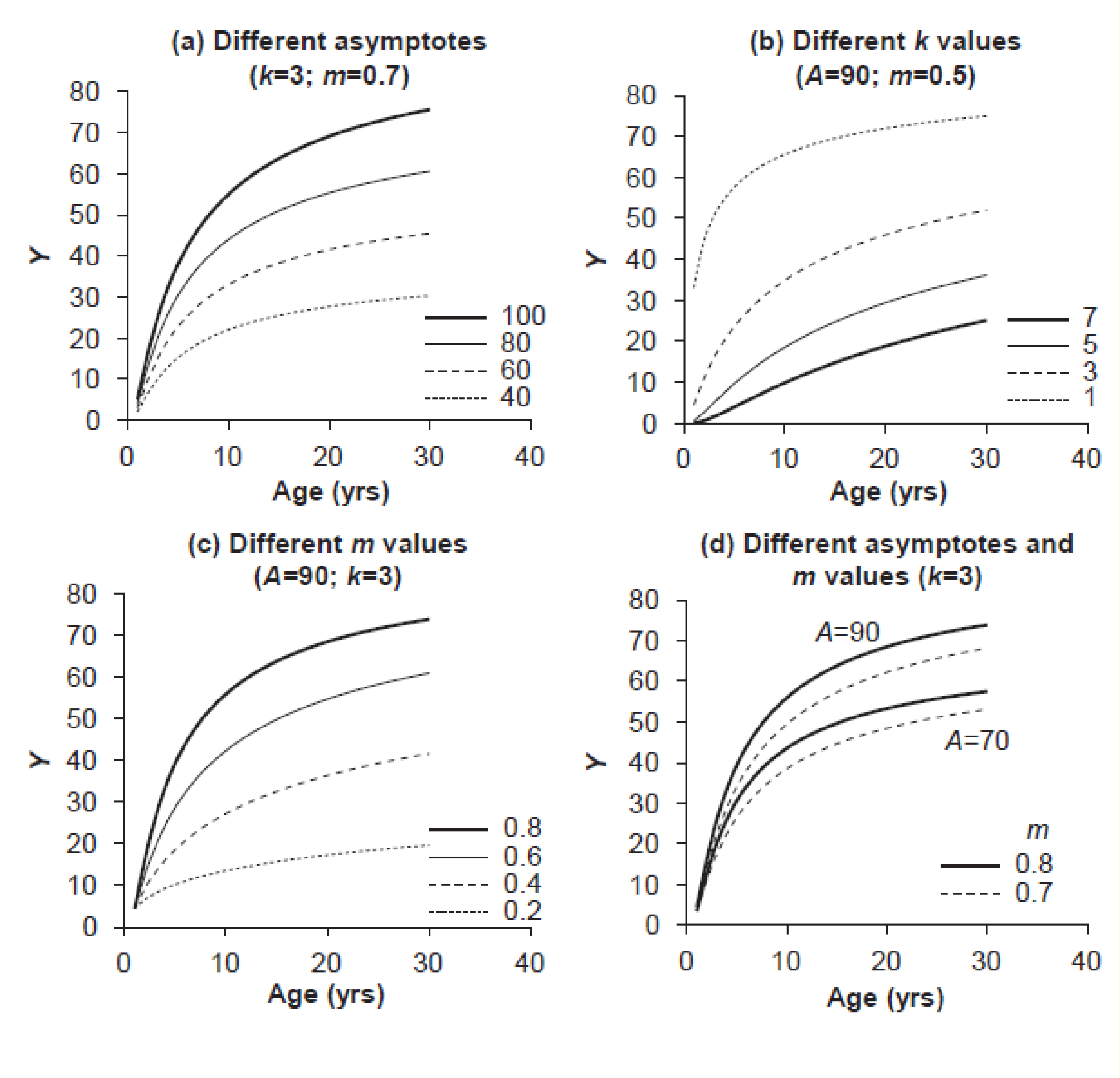
Lundqvist-Korf function

- Integral form:

$$Y = A \; e^{-k\frac{1}{t^m}}$$

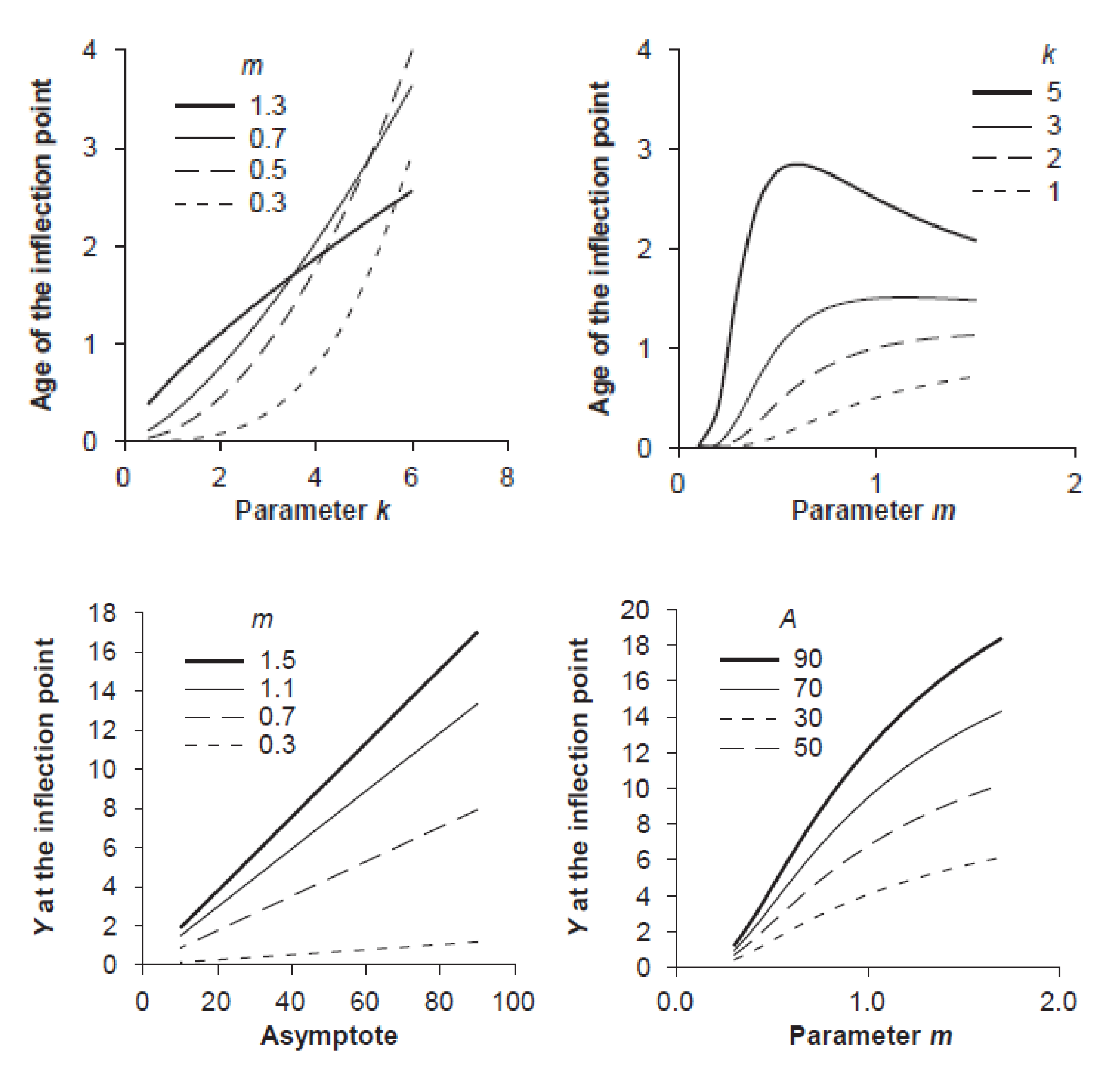
  - ✓ The *A* parameter is the assymptote
  - ✓ The *k* and *m* parameters are growth rate and shape parameters:
    - *k* is inversely related with the growth rate
    - *m* influences the age at which the inflexion point occurs

Lundqvist-Korf function



Lundqvist-Korf function

Location of the inflection point



Relationship between the functions of the Lundqvist-Korf type

- Lundqvist function
  - ✓ Schumacher's function is a specific case of Lundqvist function for m=1

Theoretical growth functions

Richards type

Monomolecular function

- Differential form
  - ✓ Assumes that the absolute growth rate is proportional to the difference between the maximum yield (asymptote) and the current yield:

$$\frac{dY}{dt} = k(A - Y)$$

Step 2: Integrate Both Sides

Integrating both sides, we have:

$$\int \frac{dY}{A - Y} = \int k\,dt$$

The left side is integrated using the rule:

$$\int \frac{dY}{A - Y} = -\ln|A - Y| + C_1$$

where C_1 is the integration constant. Thus,

$$-\ln|A - Y| = kt + C_1$$

Multiplying by -1:

$$\ln|A - Y| = -kt - C_1$$

Monomolecular function

- Differential form
  - ✓ Assumes that the absolute growth rate is proportional to the difference between the maximum yield (asymptote) and the current yield:

$$\frac{dY}{dt} = k(A - Y)$$

$$\ln|A - Y| = -kt - C_1$$

Exponentiating both sides to isolate $A - Y$:

$$|A - Y| = e^{-kt - C_1} = e^{-kt} \cdot e^{-C_1}$$

Let $e^{-C_1} = C$, a positive constant, so we have:

$$A - Y = Ce^{-kt} \quad \text{or} \quad Y - A = Ce^{-kt}$$

Since the absolute value can yield both positive and negative solutions, we consider the general solution:

$$Y(t) = A - Ce^{-kt}$$

where C is determined by initial conditions.

Monomolecular function

- Differential form
  - ✓ Assumes that the absolute growth rate is proportional to the difference between the maximum yield (asymptote) and the current yield:

$$\frac{dY}{dt} = k(A - Y)$$

Step 3: Determine C with Initial Condition

If $Y(0) = Y_0$ (initial value), then:

$$Y_0 = A - Ce^{-k\cdot 0} = A - C$$

$$C = A - Y_0$$

Thus, the final integral form of the solution is:

$$Y(t) = A - (A - Y_0)e^{-kt}$$

Monomolecular function

- Integral form:

$$Y = A\left(1 - c\, e^{-k\,t}\right)$$

$$c = e^{k\,t_0}\left(1 - \frac{Y_0}{A}\right)$$

A- assymptote;

k – shape parameter, expressing growth intensity

Logistic function

- Differential form:
  - ✓ The logistic function is based on the hypothesis that the relative growth rate is the result of the biotic potential k reduced by the current yield or size *mY* (environmental resistence):

$$\frac{1}{Y}\frac{dY}{dt} = \left(k - mY\right)$$

  - ✓ Relative growth rate is therefore a decreasing linear function of the current yield

Logistic function

- Integral form:

$$Y = \frac{A}{\left(1 + c\ e^{-k\,t}\right)}$$

$$c = \frac{Y_0/m}{k - mY_0}\ \ e^{kY_0}$$

$$A = \frac{k}{m}$$

- The inflection point occurs at *t=log(c)/k* and *Y=A/2*, which implies that the curve is symmetric around the inflection point

Generalized logistic functions

- Grosenbaugh (1965):

$$Y = \frac{A}{\left(1 + c\ e^{-\left(k_1 t + k_2 t^2 + k_3 t^3\right)}\right)}$$

- Monserud (1984)

$$Y = \frac{A}{\left(1 + c\ e^{-f\left(\boldsymbol{X}, t\right)}\right)}$$

✓ where ***X*** is a vector of several variables

Gompertz function

- Differential form:
  - ✓ This function assumes that the relative growth rate is proportional to the difference between the logarithms of the maximum yield and current yield

$$\frac{1}{Y}\frac{dY}{dt} = k\left(\log A - \log Y\right)$$

  - ✓ Which is equivalent to being inversely proportional to the logarithm of the proportion of current yield to the maximum yield

$$\frac{1}{Y}\frac{dY}{dt} = -k \log\left(\frac{Y}{A}\right)$$

Gompertz function

- Integral form:

$$Y = A\, e^{-c\, e^{-k\,t}}$$

$$c = \left(\log A - \log Y_0\right)\, e^{-k\,t_0} = \log\left(\frac{A}{Y_0}\right)\, e^{k\,t_0}$$

Richards function

- Differential form
  - ✓ The absolute growth rate of biomass (or volume) is modeled as:
    - the *anabolic rate* (construction metabolism), proportional to the photossintethicaly active area (expressed as an allometric relationship with biomass)
    - the *catabolic rate* (destruction metabolism), proportional to biomass

Anabolic rate —— $c_1 S = c_1 \left(c_0 Y^m \right) = c_2 Y^m$

Catabolic rate —— $c_3 Y$

Potential growth rate —— $c_2 Y^m - c_3 Y$

Growth rate —— $c_4 \left(c_2 Y^m - c_3 Y \right)$

S – photossintethically active biomass ; Y – biomass; m – alometric coefficient; c0,c1,c2,c3 – proportionality coefficients; c4 – eficacy coefficient

Richards function

- The differential form of the Richards function follows:

$$\frac{dY}{dt} = \eta Y^{m} - \gamma Y$$

Richards function

- Integral form:
  - ✓ By integration and using the initial condition *y(t₀)*=0, the integral form of the Richards function is obtained:

$$Y = A\left(1 - c\,e^{-k\,t}\right)^{\frac{1}{1-m}}$$

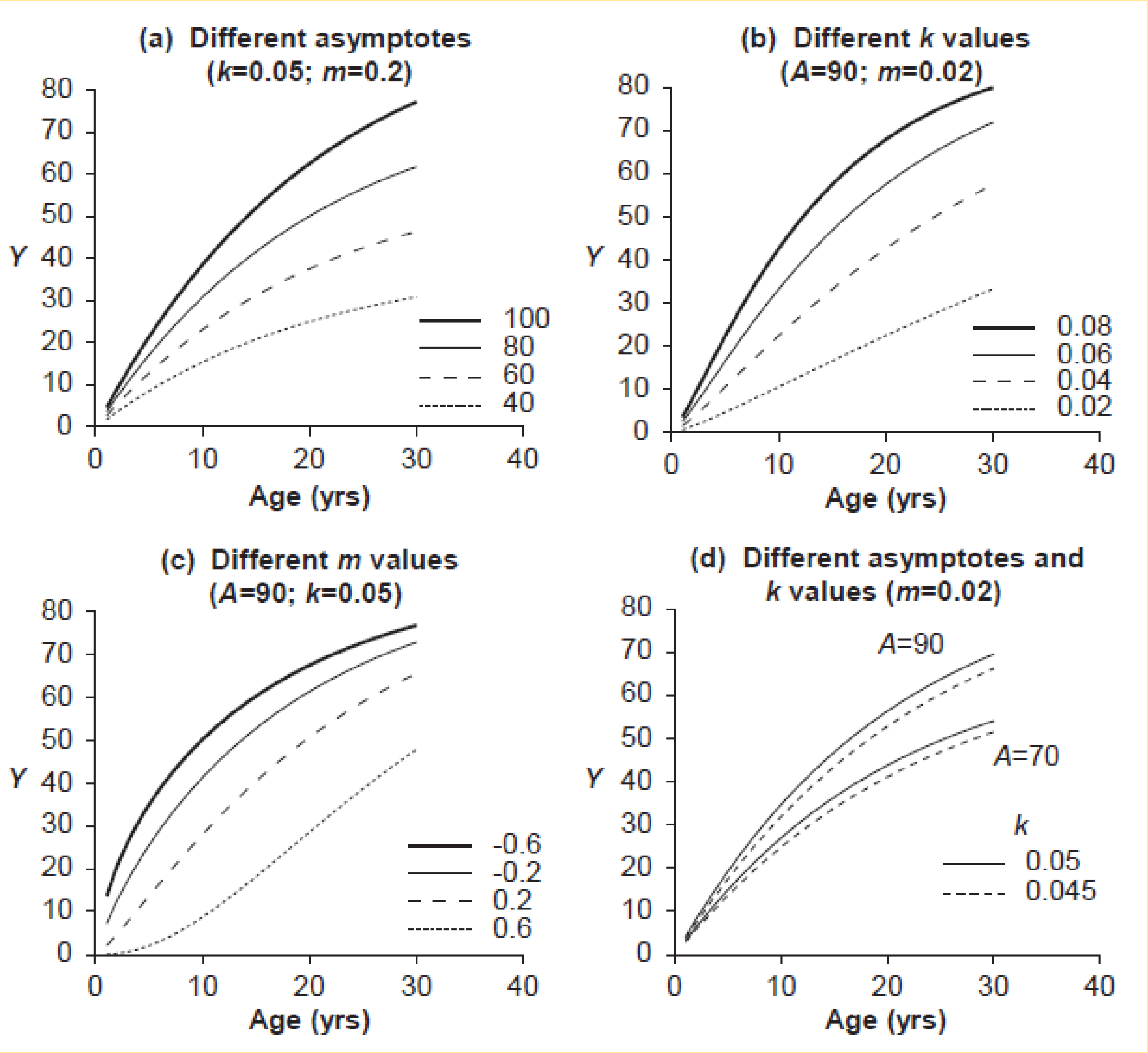
with parameters *m*, *c*, *k* and *A* where:

$$c = e^{-(1-m)\gamma\, t_0} = e^{-k\,t_0}$$

$$k = (1-m)\gamma$$

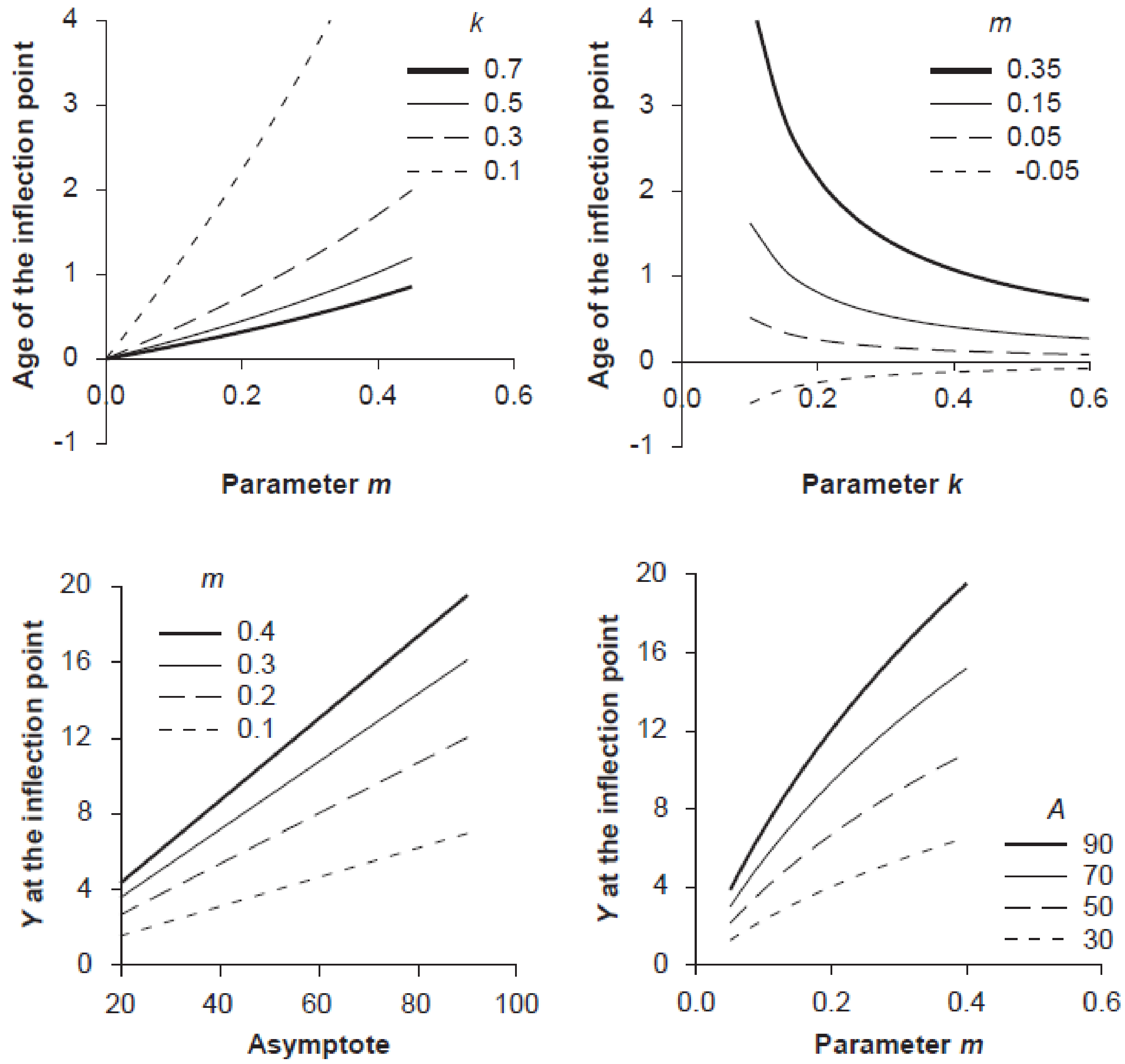
$$A = \left(\frac{\eta}{\gamma}\right)^{\frac{1}{1-m}}$$

Richards function



Richards function

Location of the inflection point



Relationship between the functions of the Richards type

- Richards function
  - ✓ Monomolecular, logistic and Gompertz are specific cases of Richards function dor the m parameter equal to 0, 2, $\rightarrow 1$

Theoretical growth functions

Hossfeld IV type

Hossfeld IV function

- The Hossfeld IV function is a sigmoid function, originally proposed in 1822 (Zeide 1993), for the description of tree growth:

$$Y = \frac{t^k}{c + t^k / A} = A\frac{t^k}{Ac + t^k} = A\frac{t^k}{c + t^k}$$

- The function can also be obtained from the generalized logistic by using *f(X,t)=-klog(t)*. Consequently some authors designate it as the log-logistic growth function

McDill-Amateis / Hossfeld IV function

- Differential form:
  - ✓ The variables considered in the development of the growth function and the respective dimensions were:

Variable	dY/dt	t	Y	A
Dimension	$L\ T^{-1}$	T	L	L

where L indicates length, T is time and A is the asymptote

  - ✓ Applying differential analysis to these variables, the following differential form is obtained:

$$\frac{dY}{dt} = k\frac{Y}{t}\left(1 - \frac{Y}{A}\right)$$

where *k* is a parameter related to the growth rate

McDill-Amateis / Hossfeld IV function

- Integral form:

$$Y = \frac{A}{1-\left(1-\frac{A}{Y_0}\right)\left(\frac{t_0}{t}\right)^k}$$

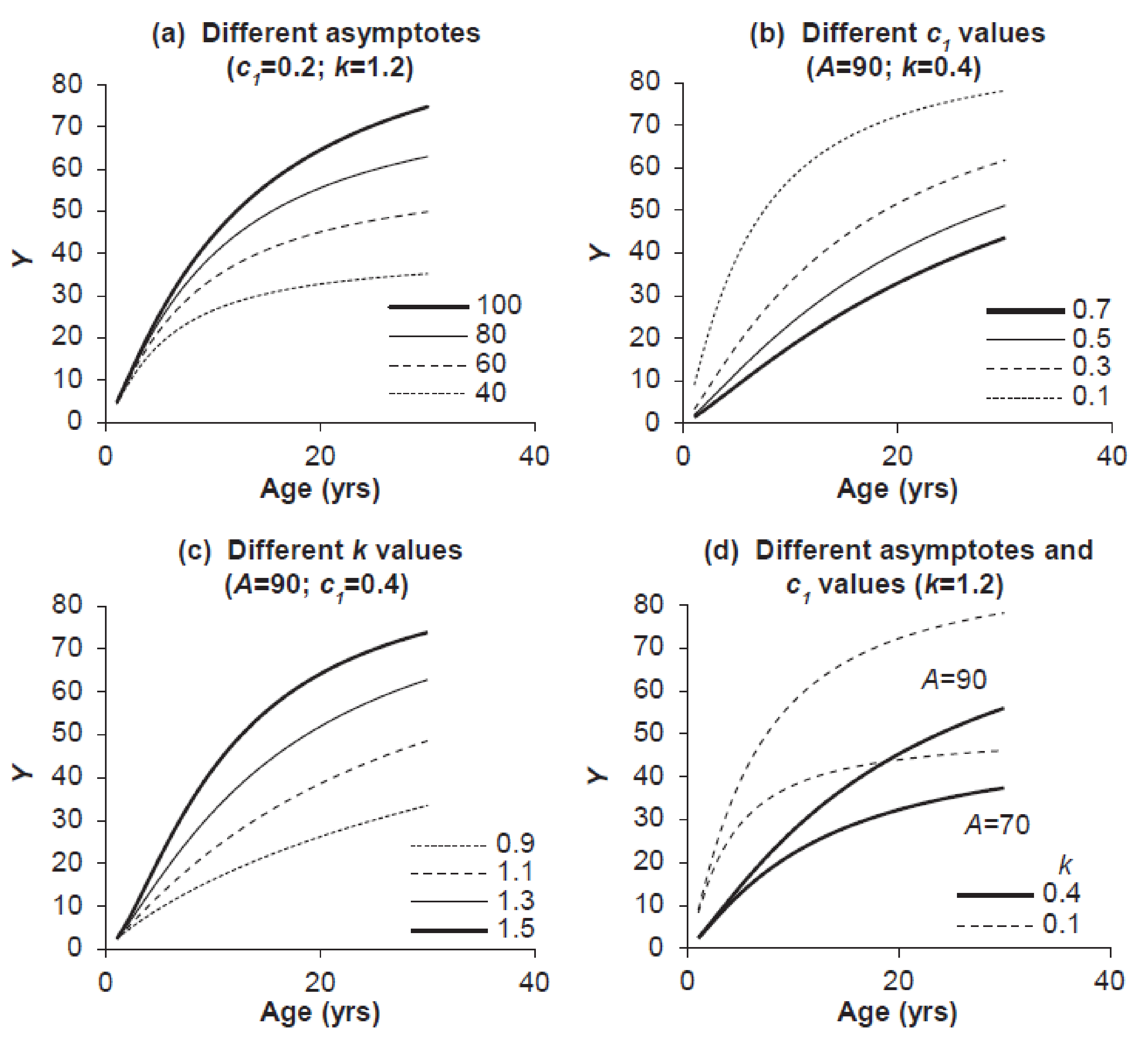
where (t_0, Y_0) is the initial condition and k expresses the growth rate

✓ By making

$$c = \left(\frac{1}{Y_0} - \frac{1}{A}\right) t_0^K$$

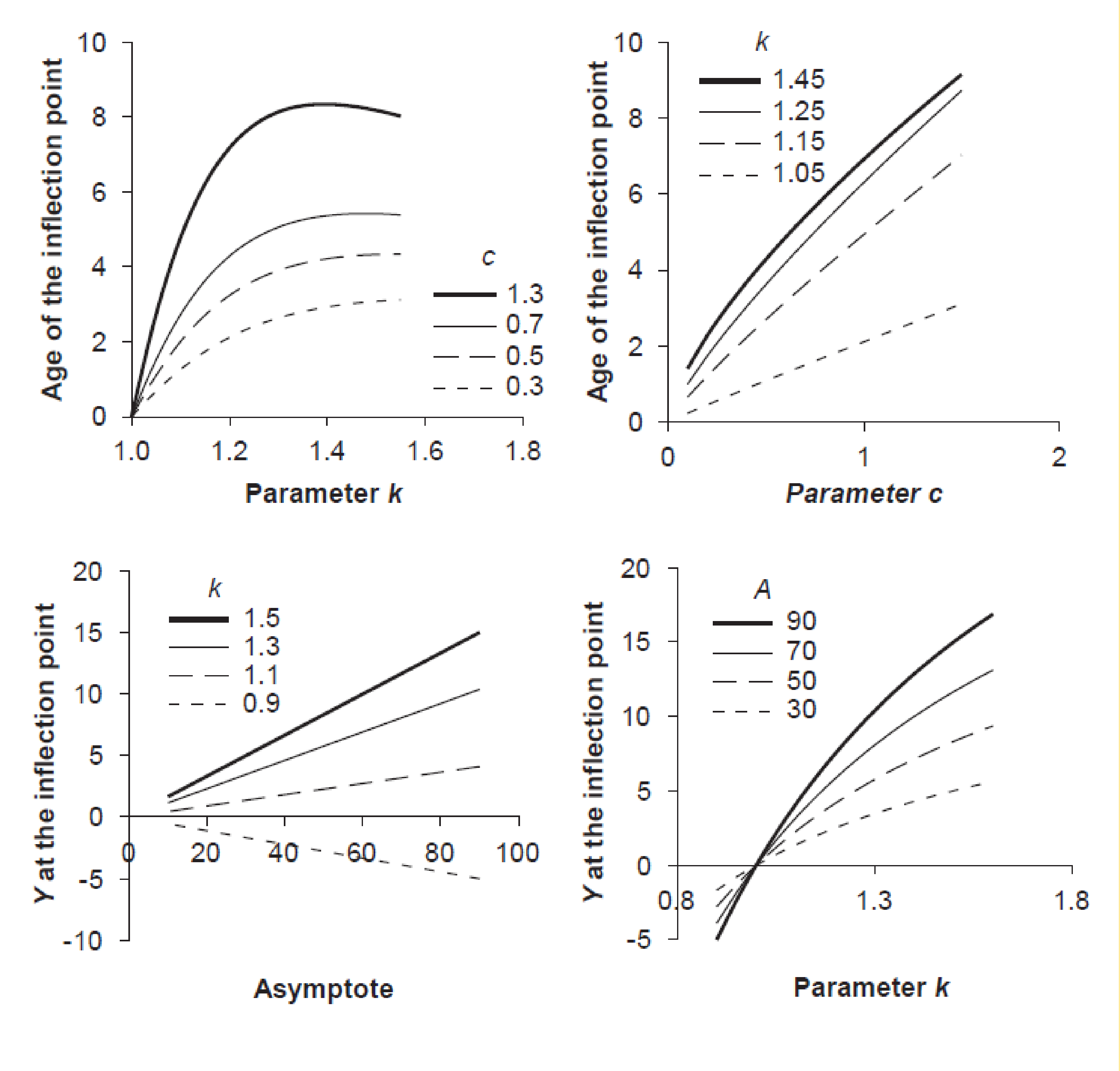
the integral form of the McDill-Amateis function coincides with the Hossfeld IV function

Hossfeld IV function



Hossfeld IV function

Location of the inflection point



(Zeide decomposition of growth functions)

Zeide decomposition of growth functions

- Zeide found out that all the growth functions can be decomposed into two components:
  - ✓ Growth expansion - represents the innate tendency towards exponential multiplication and is associated with biotic potential, photosynthetic activity, absorption of nutrients, constructive metabolism, anabolism
  - ✓ Growth decline - represents the constraints imposed by external (competition, limited resources, respiration, and stress) and internal (self-regulatory mechanisms and aging) factors

Zeide decomposition of growth functions

- The decomposition can be achieved either by a subtraction or a division (subtraction of logarithms) of the two effects
- All the equations analyzed by Zeide, except Weibull's, are particular cases of the two following forms:
- LTD $\quad ln\, y' = k + p\, ln\, y + q\, ln\, t \leftrightarrow y' = k_1 y^p t^q$
- TD $\quad ln\, y' = k + p\, ln\, y + q\, t \leftrightarrow y' = k_1 y^p e^{q\, t}$

  where $p>0$, $q<0$ and $k=e^k$
- In both forms the expansion component is proportional to ln(y) or, in the antilog form, is a power of size
- In LTD the decline component is proportional to the ln of age while in TD it is proportional to age

Zeide decomposition of growth functions

- Zeide proposed a third form in which the declining component is expressed as a function of size instead of age:

$$\ln y' = k + p \ln y + q\,y \leftrightarrow y' = k_1 y^p e^{q\,y}$$

- The three forms are very useful for the direct modeling of tree and/or stand growth – these forms provide some assurance that the resulting model will display appropriate behavior form a biological stand point

Formulating growth functions without age explicit

Formulating growth functions without age explicit

- In many applications age is not known, e.g. in trees that do not exhibit easy to measure growth rings or in uneven aged stands
- For these cases it is useful to derive formulations of growth functions in which age is not explicit
- The derivation of these formulations is obtained by expressing t as a function of the variable and the parameters and substituting it in the growth function writen for t+a (Tomé et al. 2006)

Formulating growth functions without age explicit

- **Example with the Lundqvist function**

$$Y_t = A\, e^{-k\frac{1}{t^m}} \longrightarrow t = \left[\frac{-k}{\ln(y_t/A)}\right]^{\frac{1}{m}}$$

$$Y_{t+a} = A\, e^{-k\frac{1}{(t+a)^m}} \longrightarrow Y_{t+a} = A\, e^{-k\frac{1}{\left(\left[\frac{-k}{\ln(y_t/A)}\right]^{\frac{1}{m}}+a\right)^m}}$$

Simultaneous modeling of several individuals (Families of growth functions)

Families of growth functions

- The fitting of a growth function to data from a permanent plot is starightforward

  Example:

  - ✓ Fitting the Lundqvist function to basal area and doiminant height growth data from a permanent plot

$$Y = A\,e^{-k\left(\frac{1}{t^{n}}\right)}$$

A - asymptote

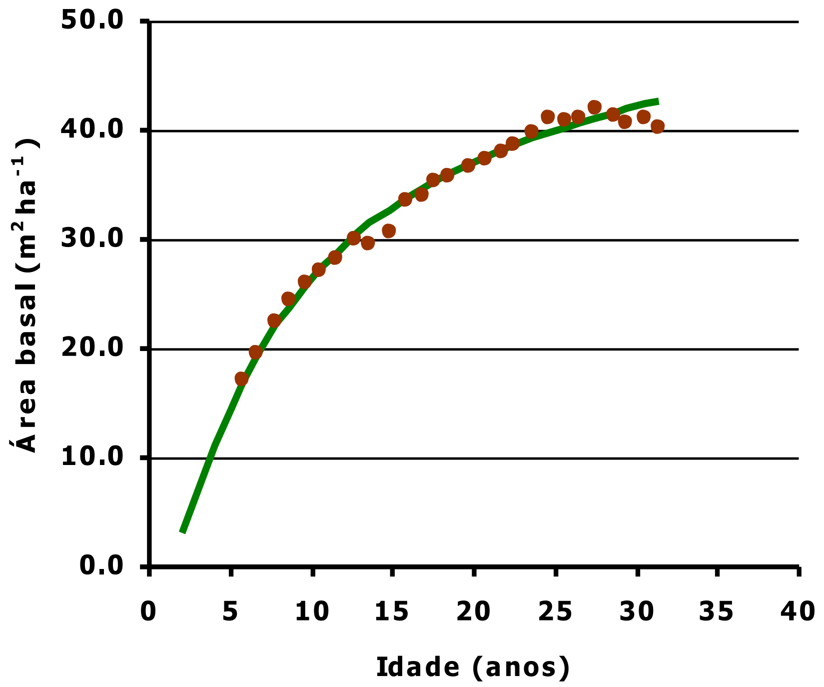
k, *n* – shape parameters

Growth functions

<u>Basal area</u>

A = 58.46, k = 5.13, n = 0.81

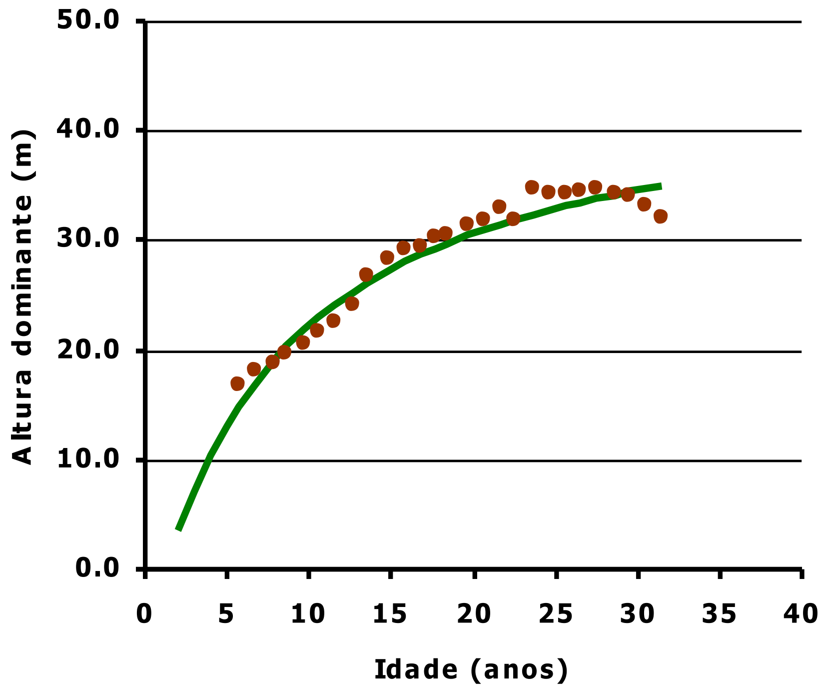
Modelling efficiency = 0.995



<u>Dominant height</u>

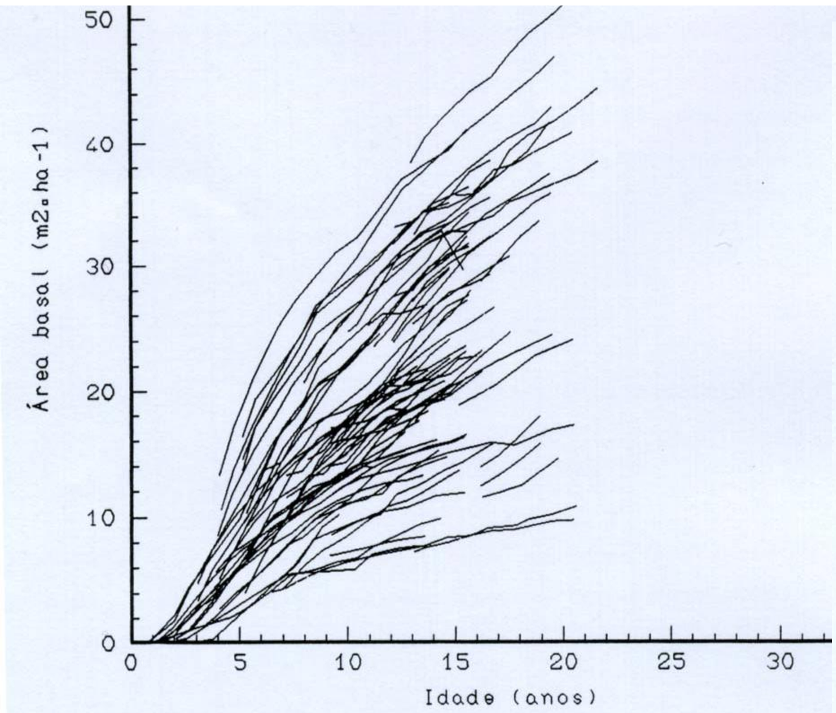
A = 48.75, k = 4.30, n = 0.75

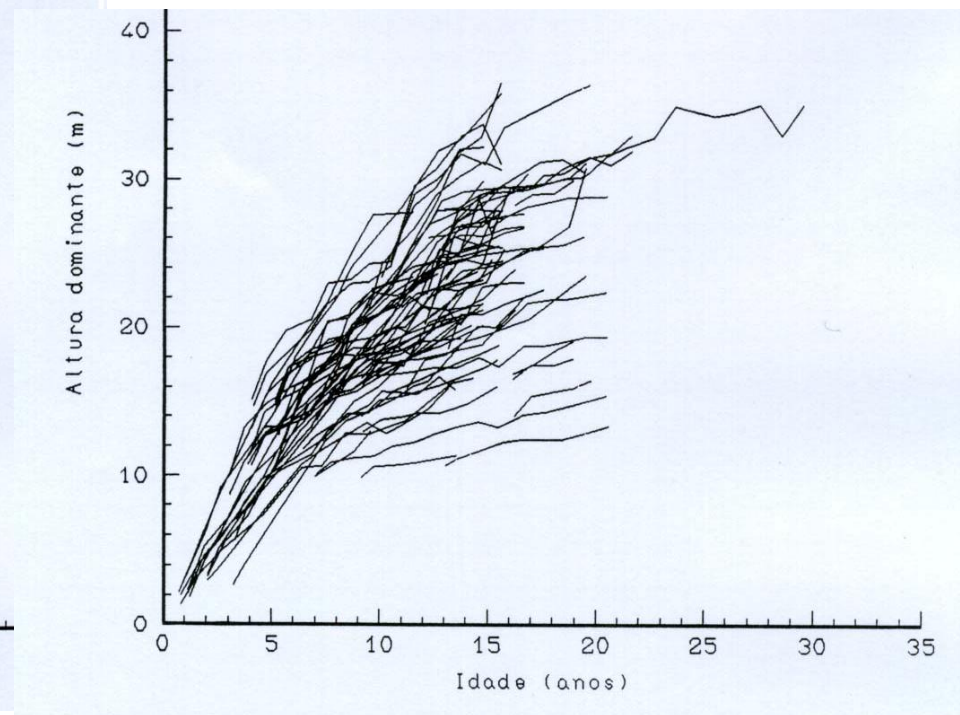
Modelling efficiency = 0.960



But how to model the growth of a series of plots?

This is our objective when developing FG&Y models…





Those plots represent “families” of curves

Using growth functions formulated as difference equations - ADA

- Algebraic difference approach (ADA)
  - ✓ When formulating a growth function as a difference equation, it is assumed that the curves belonging to the same “family” differ just by one parameter - the free parameter
  - ✓ A growth function with 3 parameters allows for 3 different formulations, usually denoted by the free parameter
  - ✓ For example for the Richards function:

    Richards-A (model with site specific asymptote)

    Richards-k (model with common asymptote)

    Ricjards-m (model with common asymptote)

Using growth functions formulated as difference equations - ADA

Example with the Lundqvist function, formulation with common asymptote and common n parameter, *A* as free parameter (Kundqvist-A):

$$Y = A\, e^{-k\frac{1}{t^m}} \Rightarrow A = \frac{Y}{e^{-k\frac{1}{t^m}}} \longrightarrow \frac{Y_2}{e^{-k\frac{1}{t_2^m}}} = \frac{Y_1}{e^{-k\frac{1}{t_1^m}}}$$

$$Y_2 = Y_1\, e^{-k\left(\frac{1}{t_2^m} - \frac{1}{t_1^m}\right)} \Leftrightarrow Y = Y_0\, e^{-k\left(\frac{1}{t^m} - \frac{1}{t_0^m}\right)}$$

A specific curve of the family is defined by the value of the free parameter

In practice, the free parameter is a function of an initial condition (Y_0,t_0)

Using growth functions formulated as difference equations - GADA

- Generalized algebraic difference approach (GADA)
  - ✓ One of the problems with ADA is the fact that it originates formulations that differ just by one parameter
  - ✓ With GADA it is possible to obtain formulations that have more than one site-specific parameter
  - ✓ In GADA parameters are assumed to be function of an unobservable set of variables (denoted by X) that expresse site differences
  - ✓ The equations is then solved by X, which, for a particular site, is substituted in the original equation (X_0)

Using growth functions formulated as difference equations - GADA

- Example with the Schumacher function

$$\ln(Y) = \alpha + \frac{\beta}{t}$$

Suppose that α=X and β=γX, then

$$\ln(Y) = X + \frac{\gamma X}{t} \longrightarrow X = \frac{\ln(Y)}{1+\gamma/t} \longrightarrow X_0 = \frac{\ln(Y_0)}{1+\gamma/t_0}$$

By substituting X0 into the previous expression, we get

$$\ln(Y) = \ln(Y_0)\frac{t_0(t-\gamma)}{t(t_0-\gamma)}$$

Using growth functions formulated as difference equations - GADA

- Another example with the Schumacher function

  Suppose now that α=X and β=X, then

$$\ln(Y) = X - \frac{\beta}{t} \quad \text{and} \quad \ln(Y) = \alpha - \frac{X}{t} \longrightarrow 2\ \ln(Y) = \left(X - \frac{\beta}{t}\right) + \left(\alpha - \frac{X}{t}\right)$$

- Solving for X:

$$X = \frac{t[\ln(Y) - \alpha] + \beta}{t - 1} \longrightarrow X_0 = \frac{t_0[\ln(Y_0) - \alpha] + \beta}{t_0 - 1}$$

- Finnally, substituting X0 in the previous expression

$$\ln(Y) = \alpha - \frac{\beta}{t} + \frac{(t-1)t_0}{(t_0 - 1)t}\left[\ln(Y_0) - \alpha + \frac{\beta}{t_0}\right]$$

Expressing parameters as a function of tree/stand variables

- Example with the Lundqvist function fit to basal area growth of eucalyptus (GLOBULUS 2.1 model) :

$$G = A_g e^{-k_g \left(\frac{1}{t}\right)^{m_g}}$$

$$A_g = A_{gQ} S^2$$

$$k_g = k_{g0} + k_{gQ} S + k_{gNp} \frac{Npl}{1000} + k_{gf} fe \quad with \;\; fe = \frac{100}{S\sqrt{Npl}}$$

$$m_g = m_{g0} + m_{gQ}\, ln(S) + m_{gNp} \frac{N}{1000}$$

Using mixed-models

- Mixed-models (linear and non-linear) “split” the model error according to different sources of variation, such as:
  - ✓ Region
  - ✓ Stand
  - ✓ Plots
  - ✓ …
- When using a model fitted with mixed-models theory it is possible to calibrate the parameters with random components by measuring a small sample of individuals
- This means that it is possible to use specific parameters for a particular tree/stand

Which is the best method to model "families" of growth functions?

- There is no best method to model "families" of growth functions
- If appropriate the three methods can be combined in order to obtain more flexible growth models

The end !!!